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Monopoles and flux strings from SU(2) adjoint scalars

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ABSTRACT: We construct, in an SU(2) gauge theory with two adjoint scalars, flux strings with monopoles attached at the ends. One scalar breaks $SU(2) \rightarrow U(1)$ and produces monopoles, the other then breaks the U(1) and produces strings. Dualizing, we write the theory in terms of effective string variables and show that the flux in the string is exactly saturated by the monopoles at the ends.

KEYWORDS: Confinement, Duality in Gauge Field Theories

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1 Introduction

Strong interactions are adequately described at high energies by quantum chromodynamics (QCD). At low energies, the QCD coupling is large and color is confined, but a precise description of how that happens is as yet unknown. Following the ideas of Mandelstam and Nambu [1, 2], it is now generally thought that the QCD vacuum behaves like a dual superconductor, created by condensation of magnetic monopoles, in which confinement is analogous to a dual Meissner effect. A meson state is then formed by attaching quarks to the ends of a flux string analogous to the Abrikosov-Nielsen-Olesen vortex string of Abelian gauge theory [3, 4].

A construction of flux strings in the Weinberg-Salam theory was suggested by Nambu [5], in which a pair of magnetic monopoles are bound by a flux string of Z condensate. A different construction of flux string, involving two adjoint scalar fields in an SU(2) gauge theory, has been discussed in [4, 6]. Recently there has been a resurgence of interest in such constructions [7–10].

In this paper we construct these strings and write the action in terms of string variables as a dual gauge theory. A pair of 't Hooft-Polyakov magnetic monopole and anti-monopole [11–13] are attached to the ends of the string, as in the picture of confinement. We start with SU(2) gauge theory with two adjoint scalars. One of them, call it ϕ_1 , acquires a vacuum expectation value (vev) \vec{v}_1 which is a vector in internal space, and breaks the symmetry group down to U(1). The 't Hooft-Polyakov monopoles are associated with this breaking. The other scalar field ϕ_2 also has a non-vanishing vev \vec{v}_2 , which is free to wind around \vec{v}_1 in the internal space. This winding is mapped to a circle in space, giving rise to the vortex string. We then dualize the fields as in [14–18] to write the action in terms of string variables.

The idea of two-scale symmetry breaking in SU(2), the first to produce monopoles and the second to produce strings, has appeared in [19]. Later this idea was used in a supersymmetric setting in [20–22]. The idea of flux matching, following Nambu [5] also appeared in these papers. In this paper we describe an explicit construction of flux strings

in non-supersymmetric SU(2) theory with 't Hooft-Polyakov monopoles of the same theory attached to the ends. We also show explicitly, by dualizing the variables, that the flux at each end of the string is saturated by the magnetic monopoles, indicating confinement of magnetic flux. As far as we are aware, this is the first time this construction has been done in terms of the variables corresponding to the macroscopic objects — the monopoles and flux tubes.

2 Magnetic monopoles

In the bulk of the paper we shall consider SU(2) gauge theory with two adjoint scalar fields. The Lagrangian for SU(2) gauge theory with two adjoint scalar can be written as

$$\mathcal{L} = -\frac{1}{2}\text{Tr} (G_{\mu\nu}G^{\mu\nu}) + \text{Tr} (D_\mu\phi_1D^\mu\phi_1) + \text{Tr} (D_\mu\phi_2D^\mu\phi_2) + V(\phi_1, \phi_2). \quad (2.1)$$

Here $\phi_\alpha = \phi_\alpha^i\tau^i$, ($\alpha = 1, 2$) are two real scalar fields in the adjoint representation of SU(2), τ^i are the SU(2) generators with $\text{Tr}(\tau^i\tau^j) = \frac{1}{2}\delta^{ij}$ ($i, j = 1, 2, 3$), and $V(\phi_1, \phi_2)$ is some interaction potential for the scalars. The covariant derivative and the Yang-Mills field strength tensor are defined as

$$D_\mu\phi_\alpha^i = \partial_\mu\phi_\alpha^i + g\epsilon^{ijk}A_\mu^j\phi_\alpha^k, \quad (2.2)$$

$$G_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + g\epsilon^{ijk}A_\mu^jA_\nu^k. \quad (2.3)$$

We will sometimes employ vector notation, in which

$$D_\mu\vec{\phi}_\alpha = \partial_\mu\vec{\phi}_\alpha + g\vec{A} \times \vec{\phi}_\alpha, \quad (2.4)$$

$$\vec{G}_{\mu\nu} = \partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu + g\vec{A}_\mu \times \vec{A}_\nu, \text{ etc.} \quad (2.5)$$

Obviously, $\vec{\phi}_1$ and ϕ_1 represent the same object. The simplest form of the potential $V(\phi_1, \phi_2)$ that will serve our purpose is

$$V(\phi_1, \phi_2) = -\frac{\lambda_1}{4}(|\phi_1|^2 - v_1^2)^2 - \frac{\lambda_2}{4}(|\phi_2|^2 - v_2^2)^2 - V_{\text{mix}}(\phi_1, \phi_2). \quad (2.6)$$

The last term $V_{\text{mix}}(\phi_1, \phi_2)$ includes all mixing terms in the potential, which involve products of the two scalar fields in some way. Through an obvious abuse of terminology, we will refer to v_1, v_2 as the vacuum expectation values of ϕ_1 and ϕ_2 , although in general the potential $V(\phi_1, \phi_2)$ will not reach local minima at v_1, v_2 . For now we will set $V_{\text{mix}}(\phi_1, \phi_2) = 0$, then v_1, v_2 will in fact define the local minima of the potential. Let us now define the ϕ_1 vacuum by

$$\begin{aligned} D_\mu\vec{\phi}_1 &= 0 \\ |\phi_1|^2 &= v_1^2. \end{aligned} \quad (2.7)$$

The vacuum expectation value of the adjoint field ϕ_1 breaks the SU(2) symmetry down to U(1) at the scale v_1 . The vector fields A^\pm and the modulus of the scalar field ϕ_1 will acquire a mass,

$$M_{A^\pm} = gv_1, \quad M_{|\phi_1|} = \sqrt{\lambda_1}v_1. \quad (2.8)$$

We assume that the vacuum expectation value v_1 is large compared to the energy scale we are interested in. This could happen, for example, if the model is derived from a supersymmetric theory in which the symmetry breaking occurs at a SUSY GUT scale. The large value of v_1 provides large masses to the gauge fields and the scalar. Well below the scale v_1 the massive gauge fields will not appear and the scalar field will be at its vacuum value v_1 . So at low energies the theory is essentially Abelian, with one massless gauge field, which we call B_μ .

At the ϕ_1 -vacuum the gauge field can be written as [23],

$$\vec{A}_\mu = B_\mu \hat{\phi}_1 - \frac{1}{g} \hat{\phi}_1 \times \partial_\mu \hat{\phi}_1, \quad (2.9)$$

where $B_\mu = \vec{A}_\mu \cdot \hat{\phi}_1$ and $\hat{\phi}_1 = \vec{\phi}_1/v_1$. Below the SU(2) breaking scale v_1 , the gauge field is defined by eq. (2.9). Looking at eq. (2.9) we can see that one of the three components of the gauge field is not fully determined after symmetry breaking. It is a massless mode, which we will call B_μ . The massive modes A_μ^\pm do not appear here because they are not excited below the scale v_1 . In a configuration with the scalar field as $\phi_1^i \rightarrow v_1 \frac{r^i}{r}$ at spatial infinity, the second term of the right hand side of eq. (2.9) corresponds to the non-Abelian gauge field for magnetic monopoles [10].

Using eq. (2.9) we can write down the field strength tensor at the ϕ_1 -vacuum,

$$\vec{G}_{\mu\nu} = \left[\partial_\mu B_\nu - \partial_\nu B_\mu + \frac{1}{g} \hat{\phi}_1 \cdot \partial_\mu \hat{\phi}_1 \times \partial_\nu \hat{\phi}_1 \right] \hat{\phi}_1 - \frac{2}{g} \partial_\mu \hat{\phi}_1 \times \partial_\nu \hat{\phi}_1. \quad (2.10)$$

A straightforward calculation shows that,

$$-\frac{1}{2} \text{Tr} (G_{\mu\nu} G^{\mu\nu}) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad (2.11)$$

where

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu - \frac{1}{g} \hat{\phi}_1 \cdot \partial_\mu \hat{\phi}_1 \times \partial_\nu \hat{\phi}_1 \quad (2.12)$$

The second term of eq. (2.12) is the ‘monopole term’. In a configuration where the scalar field at spatial infinity goes as $\phi_1^i \rightarrow v_1 \frac{r^i}{r}$, the $(ij)^{\text{th}}$ component of the second term of eq. (2.12) becomes $-\frac{\epsilon_{ijk} r^k}{gr^3}$, which we can easily identify as the field of a magnetic monopole. The flux for this monopole field is $\frac{4\pi}{g}$. If the monopole has magnetic charge Q_m , its flux is $4\pi Q_m$, and thus we find the quantization condition for unit charge, $Q_m g = 1$.

The Lagrangian at the ϕ_1 -vacuum becomes

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu \vec{\phi}_2 \cdot D^\mu \vec{\phi}_2 - \frac{\lambda_2}{4} (|\phi_2|^2 - v_2^2)^2, \quad (2.13)$$

with $F_{\mu\nu}$ given by eq. (2.12).

3 ϕ_2 string in the ϕ_1 vacuum

We started with a theory with $SU(2)$ symmetry and a pair of adjoint scalars. The non-zero vacuum expectation value of the field ϕ_1 breaks the symmetry to $U(1)$ at a scale v_1 . Below v_1 the theory is effectively an Abelian theory with magnetic monopoles. The vacuum was chosen according to eq. (2.7). However we did not fix the internal direction of the field ϕ_1 . This is because of the fact that the gauge group $SU(2)$ is transitive on the vacuum manifold S^2 and this transitivity makes the theory irrespective of the direction of ϕ_1 . There is a little group $U(1)$ in the theory which leaves ϕ_1 invariant on the vacuum and this little group becomes the remaining symmetry of the theory. Actually the little group is defined at every point on the vacuum. However, for our case the little group action is the rotation around any point on the vacuum manifold S^2 . That is why the little group is same for every point on S^2 and the little group is $U(1)$. So the remaining symmetry is also independent of the direction of ϕ_1 .

However, there is another scalar field in the theory, namely ϕ_2 . This ϕ_2 is in the adjoint representation of $SU(2)$, so it has three real scalar components. Out of the three, one component can be chosen along the direction of the field ϕ_1 . Then the other two will rotate on a two dimensional plane normal to ϕ_1 under the action of the little group $U(1)$. In other words, a gauge transformation rotates ϕ_2 around ϕ_1 . This $U(1)$ will produce flux tubes when the $U(1)$ symmetry is spontaneously broken down to Z_2 . It is natural to take the $U(1)$ breaking scale v_2 to be very small compared to the $SU(2)$ symmetry breaking scale, $v_2 \ll v_1$.

In order to find string configurations, we write the covariant derivative of ϕ_2 using eq. (2.9),

$$\begin{aligned} D_\mu \vec{\phi}_2 &= \partial_\mu \vec{\phi}_2 + g \vec{A}_\mu \times \vec{\phi}_2, \\ &= \partial_\mu \vec{\phi}_2 + g \left[B_\mu \hat{\phi}_1 - \frac{1}{g} \hat{\phi}_1 \times \partial_\mu \hat{\phi}_1 \right] \times \vec{\phi}_2, \\ &= \partial_\mu \vec{\phi}_2 + g B_\mu \hat{\phi}_1 \times \vec{\phi}_2 + \left[\hat{\phi}_1 \left(\partial_\mu \hat{\phi}_1 \cdot \vec{\phi}_2 \right) - \partial_\mu \hat{\phi}_1 \left(\hat{\phi}_1 \cdot \vec{\phi}_2 \right) \right]. \end{aligned} \quad (3.1)$$

This is of course in the ϕ_1 vacuum.

For string configurations, ϕ_2 has to approach its vacuum value at far away from the string (ϕ_1 is of course at the ϕ_1 -vacuum already). The ϕ_2 vacuum is defined by

$$|\vec{\phi}_2|^2 = v_2^2, \quad (3.2)$$

$$D_\mu \vec{\phi}_2 = 0. \quad (3.3)$$

These equations are taken in the ϕ_1 vacuum, so in particular we use eq. (3.1) in the place of eq. (3.3). If we now dot both sides of eq. (3.3) with $\hat{\phi}_1$, we get

$$\partial_\mu (\hat{\phi}_1 \cdot \vec{\phi}_2) = 0. \quad (3.4)$$

So in the ϕ_2 vacuum (which by definition is embedded in the ϕ_1 vacuum), the component of $\vec{\phi}_2$ along $\hat{\phi}_1$ remains constant.

As mentioned above, we can decompose ϕ_2 (not necessarily in the ϕ_2 vacuum) into a component along ϕ_1 and another component normal to ϕ_1 in the internal space,

$$\vec{\phi}_2 = (\hat{\phi}_1 \cdot \vec{\phi}_2) \hat{\phi}_1 + \vec{K}. \quad (3.5)$$

Then

$$D_\mu \vec{\phi}_2 = \hat{\phi}_1 \partial_\mu (\hat{\phi}_1 \cdot \vec{\phi}_2) + \partial_\mu \vec{K} - \hat{\phi}_1 (\hat{\phi}_1 \cdot \partial_\mu \vec{K}) + g B_\mu \hat{\phi}_1 \times \vec{K}, \quad (3.6)$$

and

$$(D_\mu \vec{\phi}_2)^2 = (\partial_\mu (\hat{\phi}_1 \cdot \vec{\phi}_2))^2 + g^2 (B_\mu \vec{K})^2 + (\partial_\mu \vec{K})^2 - (\hat{\phi}_1 \cdot \partial_\mu \vec{K})^2 + 2g B_\mu \hat{\phi}_1 \times \vec{K} \cdot \partial^\mu \vec{K}. \quad (3.7)$$

Since $\hat{\phi}_1 \cdot \vec{K} = 0$, we can rewrite this after a little manipulation as

$$(D_\mu \vec{\phi}_2)^2 = (\partial_\mu (\hat{\phi}_1 \cdot \vec{\phi}_2))^2 + (\partial_\mu k)^2 + k^2 \left(\hat{\kappa} \cdot \partial_\mu \hat{\kappa} \times \hat{\phi}_1 + g B_\mu \right)^2, \quad (3.8)$$

where we have written $\vec{K} = k \hat{\kappa}$. We put this expression into the action of eq. (2.13). Then in order to extract the string variables, we note that at infinite distance away from the string, ϕ_2 approaches its vacuum value $|\phi_2| \rightarrow v_2$. Further, according to eq. (3.4), $\hat{\phi}_1 \cdot \vec{\phi}_2$ also approaches a constant, so using eq. (3.5) we see that k should also approach a constant. Then the first two terms of eq. (3.8) disappear at infinity, as does the last term of eq. (2.13), and the Lagrangian at infinity behaves as

$$\mathcal{L} = \frac{k^2}{2} \left(\hat{\kappa} \cdot \partial_\mu \hat{\kappa} \times \hat{\phi}_1 + g B_\mu \right)^2 - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \quad (3.9)$$

where now k is a constant.

Since $|\vec{\phi}_2|$ and the component of $\vec{\phi}_2$ along $\hat{\phi}_1$ both approach constant values at infinity, and so does $k = |\vec{K}|$, the only degree of freedom remaining in $\vec{\phi}_2$ at infinity is an angle χ which parametrizes the rotation of $\vec{\phi}_2$ around $\hat{\phi}_1$. The first term inside the brackets in eq. (3.9) provides the $\partial_\mu \chi$ as we will see below. This is the angle which is mapped onto a circle at infinity to produce a flux string. Further, the system is in the ϕ_1 -vacuum, i.e. $\vec{\phi}_1$ is in a vacuum configuration given by eqs. (2.7) and (2.9). So in particular we can choose this vacuum to contain 't Hooft-Polyakov monopoles as discussed after eq. (2.9).

4 Monopoles and strings

With the above in mind, let us parametrize the ϕ_1 -vacuum as

$$\vec{\phi}_1 = v_1 U(\vec{x}) \tau_3 U(\vec{x})^\dagger, \quad \text{with } U(\vec{x}) \in \text{SU}(2). \quad (4.1)$$

For example, an 't Hooft-Polyakov monopole at the origin is described by

$$U = \cos \frac{\theta}{2} \begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{pmatrix} + \sin \frac{\theta}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad (4.2)$$

where $0 \leq \theta(\vec{x}) \leq \pi$ and $0 \leq \psi(\vec{x}) \leq 2\pi$ are two parameters on the group manifold. This choice of $U(\vec{x})$ leads to the field configuration

$$\vec{\phi}_1 = v_1 \frac{r^i}{r} \tau_i. \quad (4.3)$$

For this case the quantization condition will be $Q_m g = 1$. To get a monopole of higher charge, we write

$$U_n = \cos \frac{\theta}{2} \begin{pmatrix} e^{in\psi} & 0 \\ 0 & e^{-in\psi} \end{pmatrix} + \sin \frac{\theta}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad n = \pm 1, \pm 2, \pm 3, \dots \quad (4.4)$$

The integer n labels the homotopy class, $\pi_2(\text{SU}(2)/\text{U}(1)) \sim \pi_2(S^2) \sim Z$, of the scalar field configuration. Other choices of $U(\vec{x})$ can give other configurations. For example, a monopole-anti-monopole configuration [24] is given by the choice

$$U = \sin \frac{(\theta_1 - \theta_2)}{2} \begin{pmatrix} 0 & -e^{-i\psi} \\ e^{i\psi} & 0 \end{pmatrix} + \cos \frac{(\theta_1 - \theta_2)}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (4.5)$$

For our purposes, we will need to consider a ϕ_1 -vacuum configuration with $U(\vec{x}) \in \text{SU}(2)$ corresponding to a monopole-anti-monopole pair separated from each other by a distance $> 1/v_1$. Then the total magnetic charge vanishes, but each (anti-)monopole can be treated as a point particle.

We also need to choose the form of the vector $\hat{\kappa}$ as in eq. (3.8), so that it is orthogonal to $\hat{\phi}_1 = \vec{\phi}_1/v_1$ in the internal space and rotates around $\hat{\phi}_1$. Let us write

$$\hat{\kappa} \equiv \hat{\kappa}(\vec{x})^i \tau^i = e^{i\chi(\vec{x})\hat{\phi}_1(\vec{x})} U(\vec{x}) \tau_2 U^\dagger(\vec{x}) e^{-i\chi(\vec{x})\hat{\phi}_1(\vec{x})}. \quad (4.6)$$

We have used τ_2 to write $\hat{\kappa}$ here but in principle it is possible to take any constant vector orthogonal to τ_3 without affecting the results below. The $\hat{\phi}_1(\vec{x})$ used here is constructed according to eq. (4.1) with $U(\vec{x})$ as described above, and $\chi(\vec{x})$ is the angle by which the vector $U(\vec{x})\tau_2 U^\dagger(\vec{x})$ is rotated in the group.

In any case, with $\hat{\kappa}$ as in the above equation, we find

$$\hat{\kappa} \cdot \partial_\mu \hat{\kappa} \times \hat{\phi}_1 = -\partial_\mu \chi(\vec{x}) + N_\mu(\vec{x}), \quad (4.7)$$

where the vector N_μ is given by

$$N_\mu = 2i \text{Tr} \left[\partial_\mu U U^\dagger \hat{\phi}_1 \right]. \quad (4.8)$$

χ is the angle which is mapped onto a circle in space to exhibit the flux tube. As we will see now, N_μ is the (Abelian) field of the magnetic monopoles.

Let us calculate the field strength tensor for N_μ ,

$$\partial_\mu N_\nu - \partial_\nu N_\mu = -\hat{\phi}_1 \cdot \partial_\mu \hat{\phi}_1 \times \partial_\nu \hat{\phi}_1 + 2i \text{Tr} \left[(\partial_{[\mu} \partial_{\nu]} U) U^\dagger \hat{\phi}_1 \right]. \quad (4.9)$$

If we use the $U(\vec{x})$ of eq. (4.2), the first term on the right hand side of this equation is the field strength of a magnetic monopole at the origin, while the second term is a gauge dependent line singularity, commonly known as a Dirac string. In this case,

$$N_\mu = -(1 + \cos \theta) \partial_\mu \psi. \quad (4.10)$$

If θ and ψ are mapped to the polar and the azimuthal angles, N_μ is the familiar 4-potential of a magnetic monopole with a Dirac string [25]. For the $U(\vec{x})$ of the monopole-anti-monopole pair of eq. (4.5), the first term of eq. (4.9) gives the Abelian magnetic field of a monopole-anti-monopole pair, while the second term again contains a Dirac string.

The Dirac string is a red herring, and we are going to ignore it, for the following reason. The singular Dirac string appears because we have used a $U(\vec{x})$ which is appropriate for a point monopole. If we look at the system from far away, the monopoles will look like point objects, and it would seem that we should find Dirac strings attached to each of them. However, we know that the 't Hooft-Polyakov monopoles are actually not point objects, and their near magnetic field is not describable by an Abelian four-potential N_μ , so if we could do our calculations without the far-field approximation, we would not find a Dirac string.

There is another way of confirming that the Dirac string will not appear in any calculations. In the far field approximation, we have written the Lagrangian of eq. (2.13) as eq. (3.9), which we can rewrite using eq. (4.7) as

$$L = -\frac{1}{4} \left(\partial_\mu B_\nu - \partial_\nu B_\mu + \frac{1}{g} M_{\mu\nu} \right)^2 + \frac{k^2}{2} (gB_\mu - \partial_\mu \chi + N_\mu)^2, \quad (4.11)$$

where k is a constant as mentioned earlier, and $M_{\mu\nu}$ is the monopole field,

$$M_{\mu\nu} = -\hat{\phi}_1 \cdot \partial_\mu \hat{\phi}_1 \times \partial_\nu \hat{\phi}_1. \quad (4.12)$$

The second term of the Lagrangian (4.11) is the one which exhibits a flux tube or a 'physical' string (as opposed to the unphysical Dirac string, which is an artifact of the far-field approximation and can be relocated by a gauge transformation). An exactly analogous term appears in the Abelian Higgs model, where instead of k we get the physical Higgs field. This non-Abelian model also exhibits a flux string, and just like in the Abelian Higgs model, we know that the flux string here will appear along the zeroes of k , even though eq. (4.11) is written in the far field approximation, where k is a constant. The Dirac string is also an artifact of the far-field approximation, and we can get rid of it by choosing $U(\vec{x})$ such that the Dirac string lies along the zeroes of k , i.e., along the core of the flux string. Then the troublesome line singularity, which appears in the second term of eq. (4.11), is always multiplied by zero, and we can ignore it for the rest of the paper.

5 Flux tubes

In this section we dualize the effective action obtained at the end of the previous section, and express the theory in terms of the string variables. The generating functional, corresponding to the Lagrangian of eq. (4.11), can be written as

$$Z = \int \mathcal{D}B_\mu \mathcal{D}\chi \exp i \int d^4x \left[-\frac{1}{4} \left(\partial_\mu B_\nu - \partial_\nu B_\mu + \frac{1}{g} M_{\mu\nu} \right)^2 + \frac{k^2}{2} (gB_\mu - \partial_\mu \chi + N_\mu)^2 \right], \quad (5.1)$$

In the presence of flux tubes we can decompose the angle of internal rotation χ into a regular part and a singular part, $\chi = \chi^r + \chi^s$. Then χ^s measures the homotopy class and

thus corresponds to a given magnetic flux tube, and χ^r describes single valued fluctuations around this configuration. The singular part of χ is related to the world sheet Σ of the flux string according to the equation,

$$\epsilon^{\mu\nu\rho\lambda}\partial_\rho\partial_\lambda\chi^s = \Sigma^{\mu\nu}, \tag{5.2}$$

$$\Sigma^{\mu\nu} = 2\pi n \int_\Sigma d\sigma^{\mu\nu}(x(\xi)) \delta^4(x - x(\xi)), \tag{5.3}$$

where $\xi = (\xi^1, \xi^2)$ are the coordinates on the world-sheet of the flux-tube, and $d\sigma^{\mu\nu}(x(\xi)) = \epsilon^{ab}\partial_a x^\mu \partial_b x^\nu$. In the above equation 2π is the vorticity quantum in the units we are using and n is the winding number [26].

Now we have integrations over both χ^r and χ^s , and the second term in the action is now $\frac{k^2}{2}(gB_\mu - \partial_\mu\chi_s - \partial_\mu\chi_r + N_\mu)^2$. This term can be linearized by introducing an auxiliary field C_μ ,

$$\begin{aligned} & \int \mathcal{D}\chi^r \exp \left[i \int d^4x \frac{k^2}{2} (gB_\mu - \partial_\mu\chi_s - \partial_\mu\chi_r + N_\mu)^2 \right] \\ &= \int \mathcal{D}\chi^r \mathcal{D}C_\mu \exp \left[-i \int d^4x \left\{ \frac{1}{2k^2} C_\mu^2 + C^\mu (N_\mu + gB_\mu - \partial_\mu\chi^r - \partial_\mu\chi^s) \right\} \right] \\ &= \int \mathcal{D}C_\mu \delta[\partial_\mu C^\mu] \exp \left[-i \int d^4x \left\{ \frac{1}{2k^2} C_\mu^2 + C^\mu (gB_\mu + N_\mu - \partial_\mu\chi^s) \right\} \right]. \end{aligned} \tag{5.4}$$

We can resolve the constraint $\partial_\mu C^\mu = 0$ by introducing an antisymmetric tensor field $B_{\mu\nu}$ and writing C_μ in the form $C^\mu = \frac{k}{2}\epsilon^{\mu\nu\rho\lambda}\partial_\nu B_{\rho\lambda}$.

In addition, we can replace the integration over $\mathcal{D}\chi^s$ by an integration over $\mathcal{D}x_\mu(\xi)$ which represents a sum over all configurations of the world sheet of the flux tube. Here $x_\mu(\xi)$ parametrizes the surface on which the field χ is singular. The Jacobian for this change of variables gives the action for the string on the background space time [17, 27]. The string has a dynamics given by the Nambu-Goto action, plus higher order operators [28], which can be obtained from the Jacobian. We will not write the Jacobian explicitly in what follows, but of course it is necessary to include it if we want to study the dynamics of the flux tube. Integrating over the field C_μ , we get

$$\begin{aligned} Z = \int \mathcal{D}B_\mu \mathcal{D}x_\mu(\xi) \mathcal{D}B_{\mu\nu} \exp \left[i \int d^4x \left\{ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} \right. \right. \\ \left. \left. + \frac{k}{2}\Sigma_{\mu\nu}B^{\mu\nu} - \frac{k}{4}\epsilon^{\mu\nu\rho\lambda}M_{\mu\nu}B_{\rho\lambda} - \frac{kg}{2}\epsilon^{\mu\nu\rho\lambda}B_\mu\partial_\nu B_{\rho\lambda} \right\} \right], \end{aligned} \tag{5.5}$$

where we have written $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + \frac{1}{g}M_{\mu\nu}$, defined $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$, used eq. (5.2) and also written $M_{\mu\nu} = (\partial_\mu N_\nu - \partial_\nu N_\mu)$. As explained earlier, this equality holds away from the Dirac string, which can be ignored.

Let us now integrate over the field B_μ . To do this we have to linearize $F_{\mu\nu}F^{\mu\nu}$ by

introducing another auxiliary field $\chi_{\mu\nu}$,

$$\begin{aligned}
& \int \mathcal{D}B_\mu \exp \left[i \int d^4x \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{kg}{2} \epsilon^{\mu\nu\rho\lambda} B_\mu \partial_\nu B_{\rho\lambda} - \frac{k}{4} \epsilon^{\mu\nu\rho\lambda} M_{\mu\nu} B_{\rho\lambda} \right\} \right] \\
&= \int \mathcal{D}B_\mu \mathcal{D}\chi_{\mu\nu} \exp \left[i \int d^4x \left\{ -\frac{1}{4} \chi_{\mu\nu} \chi^{\mu\nu} + \frac{1}{2} \epsilon^{\mu\nu\rho\lambda} \chi_{\mu\nu} \partial_\rho B_\lambda + \frac{1}{4g} \epsilon^{\mu\nu\rho\lambda} \chi_{\mu\nu} M_{\rho\lambda} \right. \right. \\
&\quad \left. \left. - \frac{kg}{2} \epsilon^{\mu\nu\rho\lambda} B_{\mu\nu} \partial_\rho B_\lambda - \frac{k}{4} \epsilon^{\mu\nu\rho\lambda} M_{\mu\nu} B_{\rho\lambda} \right\} \right] \\
&= \int \mathcal{D}\chi_{\mu\nu} \delta \left[\epsilon^{\mu\nu\rho\lambda} \partial_\nu (\chi_{\rho\lambda} - gk B_{\rho\lambda}) \right] \\
&\quad \exp \left[i \int d^4x \left\{ -\frac{1}{4} \chi_{\mu\nu} \chi^{\mu\nu} + \frac{1}{4g} \epsilon^{\mu\nu\rho\lambda} (\chi_{\mu\nu} - kg B_{\mu\nu}) M_{\rho\lambda} \right\} \right]. \quad (5.6)
\end{aligned}$$

We can integrate over $\chi_{\mu\nu}$ by solving the δ -functional as

$$\chi_{\mu\nu} = gk B_{\mu\nu} + \partial_\mu A_\nu^m - \partial_\nu A_\mu^m. \quad (5.7)$$

Here A_μ^m is the ‘magnetic photon’, and what we have achieved is dualization of the vector potential B_μ . The result of the integration is then inserted into eq. (5.5) to give

$$\begin{aligned}
Z &= \int \mathcal{D}A_\mu^m \mathcal{D}x_\mu(\xi) \mathcal{D}B_{\mu\nu} \\
&\quad \exp \left[i \int \left\{ -\frac{1}{4} \left(gk B_{\mu\nu} + \partial_{[\mu} A_{\nu]}^m \right)^2 + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{k}{2} \Sigma_{\mu\nu} B^{\mu\nu} - j_m^\mu A^{m\mu} \right\} \right]. \quad (5.8)
\end{aligned}$$

Here $j_m^\mu = -\frac{1}{2g} \epsilon^{\mu\nu\rho\lambda} \partial_\nu M_{\rho\lambda}$ is the current of magnetic monopoles. The equations of motion for the field $B_{\mu\nu}$ and A^μ can be calculated from this to be

$$\partial_\lambda H^{\lambda\mu\nu} = -m G^{\mu\nu} - \frac{m}{g} \Sigma^{\mu\nu}, \quad (5.9)$$

$$\partial_\mu G^{\mu\nu} = j_m^\nu \quad (5.10)$$

where $G_{\mu\nu} = gk B_{\mu\nu} + \partial_\mu A_\nu^m - \partial_\nu A_\mu^m$, and $m = gk$. Combining eq. (5.10) and eq. (5.9) we find that

$$\frac{1}{g} \partial_\mu \Sigma^{\mu\nu}(x) + j_m^\nu(x) = 0. \quad (5.11)$$

It follows rather obviously that a vanishing magnetic monopole current implies $\partial_\mu \Sigma^{\mu\nu}(x) = 0$, or in other words if there is no monopole in the system, the flux tubes will be closed.

If there are monopoles at the ends of the flux tube, we need to check if the fluxes match correctly. The magnetic flux through the tube is $\frac{2n\pi}{g}$, while the total magnetic flux of the monopole is $\frac{4m\pi}{g}$, where n, m are integers. So in order for the monopoles to be confined, we must have $n = 2m$. It follows that the minimum value of n required for confinement of 't Hooft-Polyakov monopoles is $n = 2$. Clearly, if we could create a monopole-anti-monopole pair, it could break the flux tube. But we have a hierarchy of energy scales $v_1 \gg v_2$, which are respectively proportional to the mass of the monopole and the energy scale of the string. We can expect this hierarchy to prevent pair creation from string breakage.

The conservation law of eq. (5.11) also follows directly from the gauge invariance of the action in eq. (5.8) under the gauge transformations

$$\delta B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu, \quad \delta A_\mu^m = -gk\Lambda_\mu. \quad (5.12)$$

Applying this gauge transformation to the action and integrating over all Λ_μ , we can write the partition function as [18]

$$Z = \int \mathcal{D}A_\mu^m \mathcal{D}x_\mu(\xi) \mathcal{D}B_{\mu\nu} \delta \left[\frac{1}{g} \partial_\mu \Sigma^{\mu\nu} + j_m^\nu \right] \exp \left[i \int \left\{ -\frac{1}{4} (gk B_{\mu\nu} + \partial_{[\mu} A_{\nu]}^m)^2 + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{k}{2} \Sigma_{\mu\nu} B^{\mu\nu} - j_m^\mu A^{m\mu} \right\} \right]. \quad (5.13)$$

Now we can define a gauge invariant $B'_{\mu\nu} = B_{\mu\nu} + \frac{1}{gk} (\partial_\mu A_\nu^m - \partial_\nu A_\mu^m)$. The generating functional becomes,

$$Z = \int \mathcal{D}x_\mu(\xi) \mathcal{D}B'_{\mu\nu} \delta \left[\frac{1}{g} \partial_\mu \Sigma^{\mu\nu}(x) + j_m^\nu(x) \right] \exp \left[i \int \left\{ \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{1}{4} m^2 B_{\mu\nu}^2 - \frac{m}{2g} \Sigma_{\mu\nu} B'^{\mu\nu} \right\} \right]. \quad (5.14)$$

Thus these strings are analogous to the confining strings in three dimensions [29]. There is no A_μ^m , the only gauge field which is present is $B'_{\mu\nu}$. This $B'_{\mu\nu}$ field mediates the direct interaction between the confining strings.

The functional delta function at eq. (5.14) enforces that at every point of space-time, the monopole current cancels the currents of the end points of flux tube. So the monopole current must be non-zero only at the end of the flux tube. Eq. (5.14) does not carry Abelian gauge field A_μ^m , only a massive second rank tensor gauge field. All this confirms the permanent attachment of monopoles at the end of the flux tube which does not allow gauge flux to escape out of the flux tubes.

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